

# Tight Bounds for Double Coverage Against Weak Adversaries<sup>\*</sup>

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**Abstract.** We study the Double Coverage (DC) algorithm for the  $k$ -server problem in the  $(h, k)$ -setting, i.e. when DC with  $k$  servers is compared against an offline optimum algorithm with  $h \leq k$  servers. It is well-known that DC is  $k$ -competitive for  $h = k$ . We prove that even if  $k > h$  the competitive ratio of DC does not improve; in fact, it increases up to  $h + 1$  as  $k$  grows. In particular, we show matching upper and lower bounds of  $\frac{k(h+1)}{k+1}$  on the competitive ratio of DC on any tree metric.

## 1 Introduction

We consider the  $k$ -server problem defined as follows. There are  $k$  servers located on points of a metric space. In each step, a request arrives at some point of the metric space and must be served by moving some server to that point. The goal is to minimize the total distance travelled by the servers.

The  $k$ -server problem was defined by Manasse et al. [7] as a far reaching generalization of various online problems. The most well-studied of those is the paging (caching) problem, which corresponds to  $k$ -server on a uniform metric space. Sleator and Tarjan [8] gave several  $k$ -competitive algorithms for paging and showed that this is the best possible ratio for any deterministic algorithm.

Interestingly, the  $k$ -server problem does not seem to get harder on more general metrics and the celebrated  *$k$ -server conjecture* states that a  $k$ -competitive deterministic algorithm exists for every metric space. In a breakthrough result, Koutsoupias and Papadimitriou [6] showed that the work function algorithm (WFA) is  $2k - 1$  competitive for every metric space, almost resolving the conjecture. The conjecture has been settled for several special metrics (an excellent reference is [2]). In particular for the line metric, Chrobak et al. [3] gave an elegant  $k$ -competitive algorithm called Double Coverage (DC). This algorithm

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was later extended and shown to be  $k$ -competitive for all tree metrics [4]. Additionally, in [1] it was shown that WFA is  $k$ -competitive for some special metrics, including the line.

**$(h, k)$ -server problem:** In this paper, we consider the  $(h, k)$ -setting, where the online algorithm has  $k$  servers, but its performance is compared to an offline optimal algorithm with  $h \leq k$  servers. This is also known as the weak adversaries model [5], or the resource augmentation version of  $k$ -server. The  $(h, k)$ -server setting turns out to be much more intriguing and is much less understood.

For the uniform metric (the  $(h, k)$ -paging problem),  $k/(k-h+1)$ -competitive algorithms are known [8] and no deterministic algorithm can achieve a better ratio. Note that this guarantee equals  $k$  for  $h = k$ , and tends to 1 as the ratio of the number of online to offline servers  $k/h$  becomes arbitrarily large. The same competitive ratio can also be achieved for the weighted caching problem [9].

However, unlike for  $k$ -server, the underlying metric space seems to play a very important role in the  $(h, k)$ -setting. Bar-Noy and Schieber (see [2], page 175) showed that for the  $(2, k)$ -server problem on a line metric, no deterministic algorithm can be better than 2-competitive for any  $k$ . In particular, the ratio does not tend to 1 as  $k$  increases.

In fact, there is huge gap in our understanding of the problem, even for very special metrics. For example, for the line no guarantee better than  $h$  is known even when  $k/h \rightarrow \infty$ . On the other hand, the only lower bounds known are the result of Bar-Noy and Schieber mentioned above and a general lower bound of  $k/(k-h+1)$  for any metric space with at least  $k+1$  points (cf. [2] for both results). In particular, no lower bound better than 2 is known for any metric space and any  $h > 2$ , if we let  $k/h \rightarrow \infty$ . The only general upper bound is due to Koutsoupias [5], who showed that WFA is at most  $2h$ -competitive<sup>1</sup> for the  $(h, k)$ -server problem on any metric<sup>2</sup>.

**The DC algorithm:** This situation motivates us to consider the  $(h, k)$ -server problem on the line and more generally on trees. In particular, we consider the DC algorithm [3], defined as follows.

*DC-Line:* If the current request  $r$  lies outside the convex hull of current servers, serve it with the nearest server. Otherwise, we move the two servers adjacent to  $r$  towards it with equal speed until some server reaches  $r$ . If there are multiple adjacent servers at the same location, we move one of them arbitrarily.

*DC-Tree:* We move all the servers adjacent to  $r$  towards it at equal speed until some server reaches  $r$ . (Note that the set of adjacent servers can change during the move, and is constantly updated.)

There are several natural reasons to consider DC for line and trees. For paging (and weighted paging), all known  $k$ -competitive algorithms also attain the

<sup>1</sup> Actually [5] shows a slightly stronger upper bound  $\text{WFA}_k \leq 2h \text{OPT}_h - \text{OPT}_k + \text{const}$  where  $\text{OPT}_k$  and  $\text{OPT}_h$  are the optimal cost using  $k$  and  $h$  servers respectively.

<sup>2</sup> If the online algorithm knows  $h$ , it can simply disable its  $k-h$  extra servers and be  $2h-1$  competitive (which is slightly better than  $2h$ ). However, Koutsoupias (and also us) consider the setting where the online algorithm does not know  $h$ .

optimal ratio for the  $(h, k)$  version. This suggests that  $k$ -competitive algorithms for the  $k$ -server on the line might attain the “right” ratio for the  $(h, k)$ -setting. DC is the only (other than WFA) deterministic  $k$ -server algorithm known for the line and trees. Moreover, DC obtains the optimum  $k/(k-h+1)$ -competitive ratio for the  $(h, k)$ -paging problem<sup>3</sup>.

It seems plausible that WFA might perform very well for lines and trees as  $k$  increases, but no  $o(h)$  bound is known. Most known upper bounds, including [5], bound the *extended cost* instead of the actual cost of the algorithm. Using this approach we can easily show that WFA is  $(h+1)$ -competitive for the line<sup>4</sup>.

**Our Results:** We determine the exact competitive ratio of DC on lines and trees in the  $(h, k)$ -setting.

**Theorem 1.** *The competitive ratio of DC is at least  $\frac{k(h+1)}{(k+1)}$ , even for a line.*

Note that for a fixed  $h$ , the competitive ratio worsens as the number of online servers  $k$  increases! In particular, it equals  $h$  for  $k = h$  and it approaches  $h + 1$  as  $k \rightarrow \infty$ .

Consider the (seemingly trivial) case of  $h = 1$ . If  $k = 1$ , clearly DC is 1-competitive. However, for  $k = 2$  it becomes  $4/3$  competitive<sup>5</sup>. Generalizing this example to  $(1, k)$  already becomes quite involved. Our lower bound in Theorem 1 for general  $h$  and  $k$  is based on an adversarial strategy obtained by a careful recursive construction.

Next, we give a matching upper bound.

**Theorem 2.** *For any tree, the competitive ratio of DC is at most  $\frac{k(h+1)}{(k+1)}$ .*

This generalizes the previous results for  $h = k$  [3, 4]. Our proof also follows similar ideas, but our potential function is more involved (it has three terms instead of two) and the analysis is more subtle. To keep the main ideas clear, we first prove Theorem 2 for the simpler case of a line in Section 3. The proof for trees is analogous but more involved, and is described in Section 4.

## 2 Lower Bound

We now prove Theorem 1. We will describe an adversarial strategy  $S_k$  for the setting where DC has  $k$  servers and the offline optimum (adversary) has  $h$  servers

<sup>3</sup> Here, we view the uniform metric as a star graph where requests appear to the leaves.

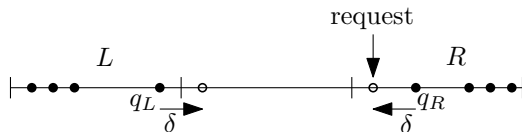
A proof of this result will be given in the full version of the paper.

<sup>4</sup> In [1] it is shown that for the line  $\text{ExtCost}_h \leq (h+1) \text{OPT}_h + \text{const}$ . Moreover in [5] the monotonicity of extended cost was proven:  $\text{ExtCost}_k \leq \text{ExtCost}_h$ . Using same arguments as in [5] it follows that  $\text{WFA}_k \leq (h+1)\text{OPT}_h - \text{OPT}_k + \text{const}$ .

<sup>5</sup> Consider the instance where all servers are at  $x = 0$  initially. A request arrives at  $x = 2$ , upon which both DC and offline move a server there and pay 2. Then a request arrives at  $x = 1$ . DC moves both servers there and pays 2 while offline pays 1. All servers are now at  $x = 1$  and the instance repeats.

and then show that the competitive ratio of DC can be made arbitrarily close to  $k(h+1)/(k+1)$ .

Roughly speaking (and ignoring some details), the strategy  $S_k$  works as follows. Let  $I = [0, b_k]$  be the *working interval* associated with  $S_k$ . Let  $L = [0, \epsilon b_k]$  and  $R = [(1-\epsilon)b_k, b_k]$  denote the (tiny) *left front* and *right front* of  $I$ . Initially, all offline and online servers are located in  $L$ . The adversary moves all its  $h$  servers to  $R$  and starts requesting points in  $R$ , until DC eventually moves all its servers to  $R$ . The strategy inside  $R$  is defined recursively depending on the number of DC servers currently in  $R$ . Roughly, if DC has  $i$  servers in  $R$ , the adversary executes the strategy  $S_i$  repeatedly inside  $R$ , until another DC server moves there, at which point it switches to the strategy  $S_{i+1}$ . When all DC servers reach  $R$ , the adversary moves all its  $h$  servers back to  $L$  and repeats the symmetric version of the above instance until all servers move from  $R$  to  $L$ . This defines a *phase*. To show the desired lower bound, we recursively bound the online and offline costs during a phase of  $S_k$  in terms of costs incurred by strategies  $S_1, S_2, \dots, S_{k-1}$ .



**Fig. 1.** DC server is pulled to the right by  $\delta$

A crucial parameter of a strategy will be the *pull*. Recall that DC moves some server  $q_L$  closer to  $R$  if and only if  $q_L$  is the rightmost DC server outside  $R$  and a request is placed to the left of  $q_R$ , the leftmost DC server in  $R$ , as shown in Figure 1. In this situation  $q_R$  moves by  $\delta$  to the left and  $q_L$  moves to the right by the same distance, and we say that the instance in  $R$  exerts a *pull* of  $\delta$  on  $q_L$ . We will be interested in the amount of pull exerted by a strategy during one phase.

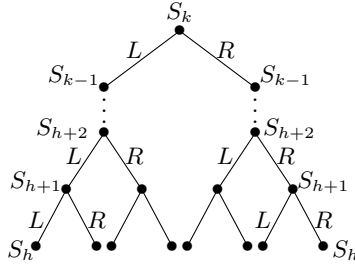
**Formal description:** We now give a formal definition of the instance. We begin by defining the following quantities associated with each strategy  $S_i$  during a single phase:

- $d_i$ , lower bound for the cost of DC inside the working interval.
- $A_i$ , upper bound for the cost of the adversary.
- $p_i, P_i$ , lower resp. upper bound for the “pull” exerted on any external DC servers located to the left of the working interval of  $S_i$ . Note that, as will be clear later, by symmetry the same pull is exerted to the right.

For  $i \geq h$ , the ratio  $r_i = \frac{d_i}{A_i}$  is a lower bound for the competitive ratio of DC with  $i$  servers against adversary with  $h$  servers.

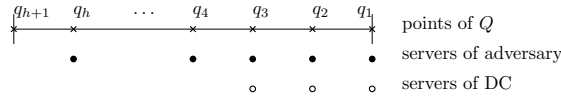
We now define the right and left front precisely. Let  $\epsilon > 0$  be a sufficiently small constant. For  $i \geq h$ , we define the size of working intervals for strategy  $S_i$

as  $s_h := h$  and  $s_{i+1} := s_i/\varepsilon$ . Note that  $s_k = h/\varepsilon^{k-h}$ . The working interval for strategy  $S_k$  is  $[0, s_k]$  and inside it we have two working intervals for strategies  $S_{k-1}$ :  $[0, s_{k-1}]$  and  $[s_k - s_{k-1}, s_k]$ . We continue this construction recursively and the nesting of these intervals creates a tree-like structure as shown in Figure 2. For  $i \geq h$ , the working intervals for strategy  $S_i$  are called type- $i$  intervals. Strategies  $S_i$ , for  $i \leq h$ , are special and are executed in type- $h$  intervals.



**Fig. 2.** Representation of strategies and the areas that they define using a binary tree.

**Strategies  $S_i$  for  $i \leq h$ :** For  $i \leq h$ , strategies  $S_i$  are performed in a type- $h$  interval (recall this has length  $h$ ). Let  $Q$  be  $h+1$  points in such an interval, with distance 1 between consecutive points.



**Fig. 3.** Strategy  $\vec{S}_3$ , where  $h \geq 3$ .

There are two variants of  $S_i$  that we call  $\vec{S}_i$  and  $\overleftarrow{S}_i$ . We describe  $\vec{S}_i$  in detail, and the construction of  $\overleftarrow{S}_i$  will be exactly symmetric. At the beginning of  $\vec{S}_i$ , we will ensure that DC servers occupy the rightmost  $i$  points of  $Q$  and offline servers occupy the rightmost  $h$  points of  $Q$  as shown in Figure 3. The adversary requests the sequence  $q_{i+1}, q_i, \dots, q_1$ . It is easily verified that DC incurs cost  $d_i = 2i$ , and its servers will return to the initial position  $q_i, \dots, q_1$ , so we can iterate  $\vec{S}_i$  again. Moreover, a pull of  $p_i = 1 = P_i$  is exerted in both directions.

For  $i < h$ , the adversary does not have to move at all, thus  $A_i = 0$ . For  $i = h$ , the offline can serve the sequence with cost  $A_h = 2$ , by using the server in  $q_h$  to serve request in  $q_{h+1}$  and then moving it back to  $q_h$ .

For strategy  $\overleftarrow{S}_i$  we just number the points of  $Q$  in the opposite direction ( $q_1$  will be leftmost and  $q_{h+1}$  rightmost). The request sequence, analysis, and assumptions about initial position are the same.

**Strategies  $S_i$  for  $i > h$ :** We define the strategy  $S_i$  for  $i > h$ , assuming that  $S_1, \dots, S_{i-1}$  are defined. Let  $I$  denote the working interval for  $S_i$ . We assume that, initially, all offline and DC servers lie in the leftmost (or analogously rightmost) type- $(i-1)$  interval of  $I$ . Indeed, for  $S_k$  this is achieved by the initial configuration, and for  $i < k$  we will ensure this condition before applying strategy  $S_i$ . In this case our phase consists of left-to-right step followed by right-to-left step (analogously, if all servers start in the rightmost interval, we apply first right-to-left step followed by left-to-right step to complete the phase).

Let  $L_j$  and  $R_j$  denote the leftmost and the rightmost type- $j$  interval contained in  $I$ , for  $h \leq j < i$ .

*Left-to-right step:*

1. Adversary moves all its servers from  $L_{i-1}$  to  $R_h$ , specifically to the points  $q_1, \dots, q_h$  to prepare for the strategy  $\overrightarrow{S}_1$ . Next, point  $q_1$  is requested which forces DC to move one server to  $q_1$  and initial conditions of  $\overrightarrow{S}_1$  are satisfied.
2. For  $j = 1$  to  $h$ : apply  $\overrightarrow{S}_j$  to interval  $R_h$  until  $(j+1)$ -th server arrives to point  $q_{j+1}$  in  $R_h$ . After server  $j+1$  arrives, we finish the already started request sequence of  $\overrightarrow{S}_j$ , so that DC servers will be lined in points  $q_{j+1}, \dots, q_1$  — ready for strategy  $\overrightarrow{S}_{j+1}$ .
3. For  $h < j < i$ : apply  $\overrightarrow{S}_j$  to interval  $R_j$  until  $(j+1)$ -th server arrives to  $R_j$ . Note that it was the only DC server moving from  $L_{i-1}$  towards  $R_j$ . The rest are either still in  $L_{i-1}$  or in  $R_j$ . Since  $R_j$  is the rightmost interval of  $R_{j+1}$  and  $L_{i-1} \cap R_{j+1} = \emptyset$ , our configuration is ready for strategy  $\overrightarrow{S}_{j+1}$ .

*Right-to-left step:* Same as Left-to-right, just replace  $\overrightarrow{S}_j$  by  $\overleftarrow{S}_j$ ,  $R_j$  intervals by  $L_j$ , and  $L_j$  by  $R_j$ .

**Bounding Costs:** We begin with a simple but useful observation that follows directly from the definition of *DC*. For any subset  $X$  of  $i \leq k$  consecutive DC servers, let us call *center of mass* of  $X$  the average position of servers in  $X$ . We call a request *external* with respect to  $X$ , when it is outside the convex hull of  $X$  and *internal* otherwise.

**Lemma 1.** *For any sequence of internal requests with respect to  $X$ , the center of mass of  $X$  remains the same.*

*Proof.* Follows trivially since for any internal request, DC moves precisely two servers by an equal amount in opposite directions.  $\square$

Let us derive values  $d_i, A_i, p_i$ , and  $P_i$  assuming that they were already computed for all  $j < i$ . We claim that the offline cost  $A_i$  for strategy  $S_i$  during a

phase can be upper bounded as follows.

$$A_i \leq 2 \left( s_i h + \sum_{j=1}^{i-1} A_j \frac{s_i}{p_j} \right) = 2s_i \left( h + \sum_{j=h}^{i-1} \frac{A_j}{p_j} \right) \quad (1)$$

The term  $2s_i h$  follows as offline initially moves the  $h$  server from left of  $I$  to right of  $I$  and then back. The costs  $A_j \frac{s_i}{p_j}$  are incurred during the phases  $S_j$  for  $j = 1, \dots, i-1$ , because  $A_j$  is an upper bound on offline cost during a phase of strategy  $S_j$  and  $\frac{s_i}{p_j}$  is an upper bound on the number of iterations of  $S_j$  during  $S_i$ . This follows because  $S_j$  (during left to right phase) executes as long as the  $(j+1)$ -th server moves from left of  $I$  to right of  $I$ . It travels distance at most  $s_i$  and feels a pull of  $p_j$  while  $S_j$  is executed in  $R$ . The equality above follows, as  $A_j = 0$  for  $j < h$ .

We now lower bound the cost of DC. Let us denote  $\delta := (1 - 2\varepsilon)$ . The length of  $I \setminus (L_{i-1} \cup R_{i-1})$  is  $\delta s_i$  and all DC servers moving from right to left have to travel at least this distance. Furthermore, as  $\frac{\delta s_i}{P_j}$  is a lower bound for the number of iterations of strategy  $S_j$ , we obtain:

$$d_i \geq 2 \left( \delta s_i i + \sum_{j=1}^{i-1} d_j \frac{\delta s_i}{P_j} \right) = 2\delta s_i \left( i + \sum_{j=1}^{i-1} \frac{d_j}{P_j} \right) \quad (2)$$

It remains to show the upper and lower bounds on the pull  $P_i$  and  $p_i$  exerted on external servers due to the (right-to-left step of) strategy  $S_i$ . Suppose  $S_i$  is executing in interval  $I$ . Let  $x$  denote the closest DC server strictly to the left of  $I$ . Let  $X$  denote the set containing  $x$  and all DC servers located in  $I$ . The crucial point is, that during the right-to-left step of  $S_i$  all requests are internal with respect to  $X$ . So by Lemma 1, the center of the mass of  $X$  stays unchanged. As  $i$  servers moved from right to left during right-to-left step of  $S_i$ , this implies that  $q$  should have been pulled to the left by the same total amount, which is at least  $i\delta s_i$  and at most  $is_i$ .

$$P_i := is_i \qquad p_i := i\delta s_i \quad (3)$$

Due to a symmetric argument, during the left-to-right step, the same amount of pull is exerted to the right.

*Proof (of Theorem 1).* The proof is by induction. In particular, for each  $i \in [h, k]$  we will show inductively that

$$\frac{d_i}{P_i} \geq 2i\delta^{i-h} \quad \text{and} \quad \frac{A_i}{p_i} \leq \frac{2(i+1)}{h+1} \delta^{-(i-h)} \quad (4)$$

Setting  $i = k$ , this implies the theorem as the competitive ratio  $r_k$  satisfies

$$r_k \geq \frac{d_k}{A_k} \geq \frac{d_k/P_k}{A_k/p_k} \geq \frac{2k}{\frac{2(k+1)}{h+1}} \frac{\delta^{k-h}}{\delta^{-(k-h)}} = \frac{k(h+1)}{k+1} \delta^{2(k-h)}$$

Choosing  $\varepsilon \ll 1/(k-h)$  small enough,  $\delta = (1 - 2\varepsilon)$  can be made arbitrarily close to 1, which implies the result.

*Induction base*  $i = h$ . For the base case we have the exact values of  $a_h$  and  $d_h$ , and, in particular,  $\frac{d_h}{P_h} = 2h$  and  $\frac{A_h}{p_h} = 2$ .

*Induction step*  $i > h$ . Using (1), (2), and (3) we obtain:

$$\frac{d_i}{P_i} = \frac{2\delta}{i} \left( i + \sum_{j=1}^{i-1} \frac{d_j}{P_j} \right) \geq \frac{2\delta}{i} \left( i + \sum_{j=1}^{i-1} 2j\delta^{j-h} \right) \geq \frac{2\delta}{i} \delta^{i-1-h} (i + i(i-1)) = 2i\delta^{i-h}$$

$$\begin{aligned} \frac{A_i}{p_i} &= \frac{2}{i\delta} \left( h + \sum_{j=h}^{i-1} \frac{A_j}{p_j} \right) \leq \frac{2}{i\delta} \left( h + \sum_{j=h}^{i-1} \frac{2(j+1)}{h+1} \delta^{-(j-h)} \right) \\ &\leq \frac{2}{i\delta} \delta^{-(i-1-h)} \left( \frac{h(h+1) + 2 \sum_{j=h}^{i-1} (j+1)}{h+1} \right) \\ &= \frac{2}{i\delta^{i-h}} \frac{i(i+1)}{h+1} = \frac{2(i+1)}{h+1} \delta^{-(i-h)} \end{aligned}$$

The last inequality follows as  $2 \sum_{j=h}^{i-1} (j+1) = i(i+1) - h(h+1)$ .  $\square$

### 3 Upper Bound

In this section, we give an algorithm that matches the lower bound from the previous section. By OPT we denote the optimal offline algorithm.

Let  $r$  be a request issued at time  $t$ . Let  $X$  denote the configuration of DC (i.e. the set of points in the line where DC servers are located) and  $Y$  the configuration of OPT before serving request  $r$ . Similarly, let  $X'$  and  $Y'$  be the corresponding configurations after serving  $r$ . In order to prove our upper bound, we define a potential function  $\Phi(X, Y)$  such that

$$DC(t) + \Phi(X', Y') - \Phi(X, Y) \leq c \cdot OPT(t), \quad (5)$$

where  $c = \frac{k(h+1)}{k+1}$  is the desired competitive ratio, and  $DC(t)$  and  $OPT(t)$  denote the cost incurred by DC and OPT at time  $t$ .

Let  $M \subseteq X$  be some fixed set of  $h$  servers of DC and  $\mathcal{M}(M, Y)$  denote the cost of the minimum weight perfect matching between  $M$  and  $Y$ . We denote

$$\Psi_M(X, Y) := \frac{k(h+1)}{k+1} \cdot \mathcal{M}(M, Y) + \frac{k}{k+1} \cdot D_M .$$

Here, for a set of points  $A$ ,  $D_A$  denotes the sum of all  $\binom{|A|}{2}$  pairwise distances between points in  $A$ . The potential function is defined as follows:

$$\begin{aligned} \Phi(X, Y) &= \min_M \Psi_M(X, Y) + \frac{1}{k+1} \cdot D_X \\ &= \min_M \left( \frac{k(h+1)}{k+1} \cdot \mathcal{M}(M, Y) + \frac{k}{k+1} \cdot D_M \right) + \frac{1}{k+1} \cdot D_X . \end{aligned}$$



Note this generalizes the potential considered in [3] for the case of  $h = k$ . In that setting, all the online servers are matched and hence  $D_M = D_X$  and is independent of  $M$ , and thus the potential above becomes  $k$  times that minimum cost matching between  $X$  and  $Y$  plus  $D_x$ . On the other hand in our setting, we need to select the right set  $M$  of DC servers to be matched to the offline servers based on minimizing  $\Psi_M(X, Y)$ .

Let us first give a useful property concerning minimizers of  $\Psi$ , which will be crucial later in our analysis. Note that  $\Psi_M(X, Y)$  is not simply the best matching between  $X$  and  $Y$ , but also includes the term  $D_M$  which makes the argument slightly subtle. We prove this lemma directly for trees, since it will be also useful in the following section.

**Lemma 2.** *Let  $X$  and  $Y$  be the configurations of DC and OPT and consider some fixed offline server at location  $y \in Y$ . There exists a minimizer  $M$  of  $\Psi$  that contains some DC server  $x$  which is adjacent to  $y$ . Moreover, there is a minimum cost matching  $\mathcal{M}$  between  $M$  and  $Y$  that matches  $x$  to  $y$ <sup>6</sup>.*

*Proof.* Let  $M'$  be some minimizer of  $\Psi_M(X, Y)$  and  $\mathcal{M}'$  be some associated minimum cost matching between  $M'$  and  $Y$ . Let  $x'$  denote the online server currently matched to  $y$  in  $\mathcal{M}'$  and suppose that  $x'$  is not adjacent to  $y$ . We denote  $x$  the adjacent server to  $y$ , in the path from  $y$  to  $x'$ .

We will show that we can always modify the matching (and  $M'$ ) without increasing the cost of  $\Phi$ , so that  $y$  is matched to  $x$ . We consider two cases depending on whether  $x$  is matched or unmatched.

1. If  $x \in M'$ : Let us call  $y'$  the offline server which is matched to  $x$  in  $M'$ . We swap the edges and match  $x$  to  $y$  and  $x'$  to  $y'$ . The cost of the edge connecting  $y$  in the matching reduces by exactly  $d(x', y) - d(x, y) = d(x', x)$ . On the other hand, the cost of the matching edge for  $y'$  increases by  $d(x', y') - d(x, y') \leq d(x, x')$ . Thus, the new matching has no larger cost. Moreover, the set of matched servers  $M = M'$  and hence  $D_M = D_{M'}$ , which implies that  $\Psi_M(X, Y) \leq \Psi_{M'}(X, Y)$ .
2. If  $x \notin M'$ : In this case, we set  $M = M' \setminus \{x'\} \cup \{x\}$  and we form  $\mathcal{M}$ , where  $y$  is matched to  $x$  and all other offline servers are matched to the same server as in  $\mathcal{M}'$ . Now, the cost of the matching reduces by  $d(x', y) - d(x, y) = d(x, x')$  and  $D_M \leq D_{M'} + (h-1) \cdot d(x, x')$  (as the distance of each server in  $M' \setminus \{x'\}$  to  $x$  can be greater than the distance to  $x'$  by at most  $d(x, x')$ ). This gives

$$\begin{aligned} \Psi_M(X, Y) - \Psi_{M'}(X, Y) &\leq -\frac{(h+1)k}{k+1} \cdot d(x, x') + \frac{k(h-1)}{k+1} \cdot d(x, x') \\ &= -\frac{2k}{k+1} \cdot d(x, x') < 0 \quad , \end{aligned}$$

and hence  $\Psi_M(X, Y)$  is strictly smaller than  $\Psi_{M'}(X, Y)$ .

<sup>6</sup> We remark that this property does not hold (simultaneously) for every offline server, but only for a single fixed offline server  $y$ .

□

We are now ready to prove Theorem 2 for the line.

*Proof.* Recall, that we are at time  $t$  and request  $r$  is arriving. We divide the analysis into two steps: (i) the offline serves  $r$  and then (ii) the online serves it. As a consequence, whenever a server of DC serves  $r$ , we can assume that a server of OPT is already there.

For all following steps considered,  $M$  will be the minimizer of  $\Psi_M(X, Y)$  in the beginning of the step. It might happen that, after change of  $X, Y$  during the step, better minimizer can be found. However, upper bound for  $\Delta\Psi_M(X, Y)$  is sufficient to bound the change in the first term of the potential function.

*Offline moves:* If offline moves one of its servers by distance  $d$  to serve  $r$  the value of  $\Psi_M(X, Y)$  increases by at most  $\frac{k(h+1)}{k+1}d$ . As  $OPT(t) = d$  and  $X$  does not change, it follows that

$$\Delta\Phi(X, Y) \leq \frac{k(h+1)}{k+1} \cdot OPT(t) ,$$

and hence (5) holds. We now consider the second step when DC moves.

*DC moves:* We consider two cases depending on whether DC moves a single server or two servers.

1. Suppose DC moves its rightmost server (the leftmost server case is identical) by distance  $d$ . Let  $y$  denote the offline server at  $r$ . By Lemma 2 we can assume that  $y$  is matched to the rightmost server of DC. Thus, the cost of the minimum cost matching between  $M$  and  $Y$  decreases by  $d$ . Moreover  $D_M$  increases by exactly  $(h-1)d$  (as the distance to rightmost server increases by  $d$  for all servers of DC). Thus,  $\Psi_M(X, Y)$  changes by

$$-\frac{k(h+1)}{k+1} \cdot d + \frac{k(h-1)}{k+1} \cdot d = -\frac{2k}{k+1} \cdot d .$$

Similarly,  $D_X$  increases by exactly  $(k-1)d$ . This gives us that

$$\Delta\Phi(X, Y) \leq -\frac{2k}{k+1} \cdot d + \frac{k-1}{k+1} \cdot d = -d .$$

As  $DC(t) = d$ , this implies that (5) holds.

2. We now consider the case when DC moves 2 servers  $x$  and  $x'$ , each by distance  $d$ . Let  $y$  denote the offline server at the request  $r$ . By Lemma 2 applied to  $y$ , we can assume that  $M$  contains at least one of  $x$  or  $x'$ , and that  $y$  is matched to one of them (say  $x$ ) in some minimum cost matching  $\mathcal{M}$  of  $M$  to  $Y$ . We note that  $D_X$  decreases by precisely  $2d$ . In particular, the distance between  $x$  and  $x'$  decreases by  $2d$ , and for any other server of  $X \setminus \{x, x'\}$  its total distance to other servers does not change. Moreover,  $DC(t) = 2d$ . Hence, to prove (5), it suffices to show

$$\Delta\Psi_M(X, Y) \leq -\frac{k}{k+1} \cdot 2d . \tag{6}$$

To this end, we consider two sub-cases.

- (a) *Both  $x$  and  $x'$  are matched:* In this case, the cost of the matching  $\mathcal{M}$  does not go up as the cost of the matching edge  $(x, y)$  decreases by  $d$  and the move of  $x'$  can increase the cost of matching by at most  $d$ . Moreover,  $D_M$  decreases by precisely  $2d$  (due to  $x$  and  $x'$  moving closer). Thus,  $\Delta\Psi_M(X, Y) \leq -\frac{k}{k+1} \cdot 2d$ , and hence (6) holds.
- (b) *Only  $x$  is matched (to  $y$ ) and  $x'$  is unmatched:* In this case, the cost of the matching  $\mathcal{M}$  decreases by  $d$ . Moreover,  $D_M$  can increase by at most  $(h-1)d$ , as  $x$  can move away from each server in  $M \setminus \{x\}$  by distance at most  $d$ . So

$$\Delta\Psi_M(X, Y) \leq -\frac{(h+1)k}{k+1} \cdot d + \frac{k(h-1)}{k+1} \cdot d = -\frac{2k}{k+1} \cdot d ,$$

i.e., (6) holds. □

## 4 Extension to Trees

We now consider tree metrics. Specifically, we prove Theorem 2. Part of the analysis carries over from the previous section. We use the same potential function as for the line. Observe that Lemma 2 holds for trees: We only used the triangle inequality and the fact that there exists a unique path between any two points.

*Proof (of Theorem 2).* The analysis of the step when offline moves is exactly the same as for the line. In particular, if the offline algorithm moves by distance  $d$ , only the matching cost is affected in the potential function and it can increase by at most  $d \cdot k(h+1)/(k+1)$ .

It remains to analyze the change in the potential caused by the moves of DC. In that case, we break down the DC move into *elementary moves*. Let us call *active* the servers adjacent to the requested point  $r$ , i.e., the ones which are moving. An elementary move ends when any server reaches either the request  $r$  or a vertex of the tree. In the latter case, another elementary move immediately follows, perhaps with a different set of active servers. We are going to prove that (5) holds for every elementary move. By summation, this implies that it holds for the entire DC move.

Consider an elementary move where  $q$  servers are moving by distance  $d$ . We need to establish some notation first: Let  $M$  be a minimizer of  $\Psi_M(X, Y)$  at the beginning of the step and  $A$  be the set of active servers. Let us imagine for now, that the requested point  $r$  is the root of the whole tree. For  $a \in A$  let  $Q_a$  denote the set of DC servers in the subtree below  $a$  (but including  $a$ ). We set  $q_a := |Q_a|$  and  $h_a := |Q_a \cap M|$ . Finally, let  $A_M := A \cap M$ .

By Lemma 2, we can assume that one of the active servers is matched to offline server in  $r$ . We get that  $\mathcal{M}(M, Y)$  increases by at most  $(|A_M| - 2) \cdot d$ .

In order to calculate the change in  $D_X$  and  $D_M$ , it is convenient to consider the moves of active servers sequentially rather than simultaneously.

For  $D_X$ , it is clear that each  $a \in A$ , moves further away from  $q_a - 1$  DC servers by distance  $d$  and gets closer to  $k - q_a$  by the same distance. Thus, the change of  $D_X$  associated with  $a$  is  $(q_a - 1 - (k - q_a))d = (2q_a - k - 1)d$ . Overall,

$$\Delta D_X = \sum_{a \in A} (2q_a - k - 1)d = (2k - q(k + 1))d, \quad \text{as } \sum_{a \in A} q_a = k.$$

Similarly, for  $D_M$ , we first note that it can change only due to moves of servers in  $A_M$ . Specifically, each  $a \in A_M$ , moves further away from  $h_a - 1$  matched DC servers and gets closer to the rest  $h - h_a$  of them. Thus, the change of  $D_M$  associated with  $a$  is  $(2h_a - h - 1)d$ , so overall we have

$$\Delta D_M = \sum_{a \in A_M} (2h_a - h - 1)d \leq (2h - |A_M|(h + 1))d,$$

as  $\sum_{a \in A_M} h_a \leq \sum_{a \in A} h_a = h$ .

Using above inequalities, we see that the change of potential is at most

$$\begin{aligned} \Delta \Phi(X, Y) &\leq \frac{d}{k+1} (k(h+1)(|A_M| - 2) + k(2h - |A_M|(h+1)) + (2k - q(k+1))) \\ &= \frac{d}{k+1} (-2k(h+1) + 2kh + (2k - q(k+1))) \\ &= \frac{d}{k+1} (-q(k+1)) = -q \cdot d, \end{aligned}$$

As the cost of DC is  $q \cdot d$ , we get that (5) holds, which completes the proof.  $\square$

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